

Stability of Salpeter Solutions

Wolfgang LUCHA* and Franz F. SCHÖBERL†

**Institute for High Energy Physics, Austrian Academy of Sciences,
Nikolsdorfergasse 18, A-1050 Vienna, Austria*

†*Faculty of Physics, University of Vienna, Boltzmannngasse 5, A-1090 Vienna, Austria*

Abstract. By analytical spectral analysis the bound-state solutions of the reduced Salpeter equation with harmonic-oscillator interaction are shown to be free of the instabilities numerically found in the (full) Salpeter equation for confining Bethe–Salpeter interaction kernels of certain Lorentz structure.

Keywords: Bethe–Salpeter formalism, three-dimensional reduction, instantaneous approximation, Salpeter equation, reduced Salpeter equation, bound states in quantum field theory, stability analysis

PACS: 11.10.St, 03.65.Ge, 03.65.Pm

INTRODUCTION: MOTIVATION

In the framework of ‘instantaneous approximations’ to the Bethe–Salpeter formalism for the description of bound states within quantum field theories, depending on the Lorentz structure of the Bethe–Salpeter interaction kernel solutions of the (full) Salpeter equation with some confining interactions may exhibit certain instabilities [1], possibly related to the Klein paradox, signalling the decay of states assumed to be bound by these confining interactions, and observed in *numerical* (variational) studies [1] of the Salpeter equation.

The (presumably) simplest scenario allowing for the *fully analytic* investigation of this problem is set by the *reduced* Salpeter equation [2] with *harmonic-oscillator* interaction. In this case, the integral equation of Salpeter simplifies to either an algebraic relation or a second-order homogeneous linear ordinary differential equation, immediately accessible to standard techniques. There one can hope to be able to decide unambiguously whether this setting poses a well-defined (eigenvalue) problem the solutions of which correspond to *stable* bound states associated to *real* energy eigenvalues that are *bounded from below*.

REDUCED SALPETER EQUATION FOR CONFINING INTERACTION KERNELS OF HARMONIC-OSCILLATOR TYPE

Assuming, as usual, the Lorentz structures of the effective couplings of both fermion and antifermion to be represented by identical matrices in Dirac space, generically labeled Γ , and denoting the associated Lorentz-scalar interaction function by $V_\Gamma(\mathbf{p}, \mathbf{q})$, the so-called *reduced Salpeter equation* [2] describing bound states composed of some fermion and its corresponding antifermion (with common mass m and relative momentum \mathbf{p}) reads for a bound state with mass eigenvalue M in the center-of-momentum frame of the bound state

$$(M - 2E) \Phi(\mathbf{p}) = \Lambda^+(\mathbf{p}) \gamma_0 \int \frac{d^3q}{(2\pi)^3} \sum_\Gamma V_\Gamma(\mathbf{p}, \mathbf{q}) \Gamma \Phi(\mathbf{q}) \Gamma \Lambda^-(\mathbf{p}) \gamma_0, \quad (1)$$

with the one-particle kinetic energy E and energy projectors $\Lambda^\pm(\mathbf{p})$ defined according to

$$E \equiv \sqrt{p^2 + m^2}, \quad p \equiv |\mathbf{p}| \equiv \sqrt{\mathbf{p}^2}, \quad \text{and} \quad \Lambda^\pm(\mathbf{p}) \equiv \frac{E \pm \gamma_0(\boldsymbol{\gamma} \cdot \mathbf{p} + m)}{2E}.$$

Any solution to this eigenvalue problem is the Salpeter amplitude $\Phi(\mathbf{p})$ of a bound state. Let the Bethe–Salpeter kernel be of convolution type, $V_\Gamma(\mathbf{p}, \mathbf{q}) = V_\Gamma(\mathbf{p} - \mathbf{q})$, arising from a central potential $V(r)$, $r \equiv |\mathbf{x}|$, in configuration space. Then, for the harmonic-oscillator potential $V(r) = ar^2$, $a = a^* \neq 0$, the reduced Salpeter equation becomes a second-order differential equation utilizing the Laplacian acting on states of angular momentum $\ell = 0$,

$$D \equiv \frac{d^2}{dp^2} + \frac{2}{p} \frac{d}{dp}.$$

In order to make contact with related previous analyses [3–6], let us present our line of argument for fermion–antifermion bound states of total spin J , parity $P = (-1)^{J+1}$, and charge-conjugation quantum number $C = (-1)^J$, called 1J_J spectroscopically. Due to the projectors $\Lambda^\pm(\mathbf{p})$ on the right-hand side of the reduced Salpeter equation (1), the Salpeter amplitudes $\Phi(\mathbf{p})$ describing these states contain only *one independent component* $\phi(\mathbf{p})$:

$$\Phi(\mathbf{p}) = 2\phi(\mathbf{p})\Lambda^+(\mathbf{p})\gamma_5.$$

Somewhat more specifically, we consider *pseudoscalar* (1S_0) bound states, characterized by the spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$. A brief moment of thought reveals that the cumbersome instabilities should appear first in the pseudoscalar channel. Stripping off all dependence on angular variables [7] converts this “*harmonic-oscillator reduced Salpeter equation*” into the eigenvalue equation of a (Hamiltonian) operator \mathcal{H} :

$$\mathcal{H}\phi(p) = M\phi(p). \quad (2)$$

It is a straightforward task to work out the explicit form of all the Hamiltonians \mathcal{H} for the most popular choices of the Lorentz structure of Bethe–Salpeter kernels (cf. Table 1).

TABLE 1. Hamiltonian (differential or mere multiplication) operators \mathcal{H} entering into the eigenvalue equation (2) equivalent to the reduced Salpeter equation (1) with harmonic-oscillator interaction potential $V(r) = ar^2$, for several frequently considered Lorentz structures of the Bethe–Salpeter interaction kernel.

Lorentz structure	$\Gamma \otimes \Gamma$	\mathcal{H}
Lorentz scalar	$1 \otimes 1$	$2E + a \left(\frac{2p^2 + 3m^2}{2E^4} + \frac{m^2}{E} D \frac{1}{E} \right)$
time-component Lorentz vector	$\gamma^0 \otimes \gamma^0$	$2E + a \left(\frac{2p^2 + 3m^2}{2E^4} - D \right)$
Lorentz vector	$\gamma_\mu \otimes \gamma^\mu$	$2E + a \left(\frac{m^2}{E} D \frac{1}{E} - 2D \right)$
Lorentz pseudoscalar	$\gamma_5 \otimes \gamma_5$	$2E + a \frac{2p^2 + 3m^2}{2E^4}$
Böhm–Joos–Krammer (BJK) [8]	$\frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$	$2E - aD$

SPECTRAL PROPERTIES OF THE REDUCED-SALPETER HAMILTONIAN OPERATORS \mathcal{H}

The spectra of the multiplication or differential operators \mathcal{H} for various Dirac structures $\Gamma \otimes \Gamma$ fixed by the Bethe–Salpeter kernel exhibit the following stability-relevant features:

- All of our *Hamiltonian operators* \mathcal{H} are *self-adjoint* since the differential operators D and $m^2 E^{-1} D E^{-1}$ as well as the multiplication by any real-valued function define self-adjoint operators. Consequently, the entire *spectrum* of any operator \mathcal{H} is *real*. For “reasonable” interaction kernels, that is, for kernels only constructed in terms of Dirac matrices Γ subject to $\gamma_0 \Gamma^\dagger \gamma_0 = \pm \Gamma$ and potential functions $V_\Gamma(\mathbf{p}, \mathbf{q})$ subject to $V_\Gamma^*(\mathbf{q}, \mathbf{p}) = V_\Gamma(\mathbf{p}, \mathbf{q})$, the *reality* of all *eigenvalues* M is guaranteed by the relation [7]

$$M \int \frac{d^3 p}{(2\pi)^3} \text{Tr} [\Phi^\dagger(\mathbf{p}) \Phi(\mathbf{p})] = 2 \int \frac{d^3 p}{(2\pi)^3} E \text{Tr} [\Phi^\dagger(\mathbf{p}) \Phi(\mathbf{p})] + \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \sum_\Gamma V_\Gamma(\mathbf{p}, \mathbf{q}) \text{Tr} [\Phi^\dagger(\mathbf{p}) \gamma_0 \Gamma \Phi(\mathbf{q}) \Gamma \gamma_0],$$

satisfied by any Salpeter amplitude $\Phi(\mathbf{p})$ solving the reduced Salpeter equation (1).

- For the Lorentz *pseudoscalar*, $\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$, and, if $m = 0$, for the Lorentz *scalar*, $\Gamma \otimes \Gamma = 1 \otimes 1$, our Hamiltonians \mathcal{H} form pure multiplication operators, with purely *continuous* spectrum. Bound states do not exist, stability questions thus do not arise.
- For the *time-component* Lorentz *vector*, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, for the Lorentz structure introduced in [8], $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$, and, if $m = 0$, for the Lorentz *vector*, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, our Hamiltonians \mathcal{H} form ($\ell = 0$) Schrödinger operators with a positive, infinitely rising potential $V(p) \rightarrow \infty$ for $p \rightarrow \infty$, provided, of course, the sign of the coupling a is chosen appropriately. The latter operators have entirely *discrete* spectra bounded from below; all bound states may be expected to be *stable*. Figure 1 illustrates the typical qualitative behaviour of the effective potentials $V(p)$.

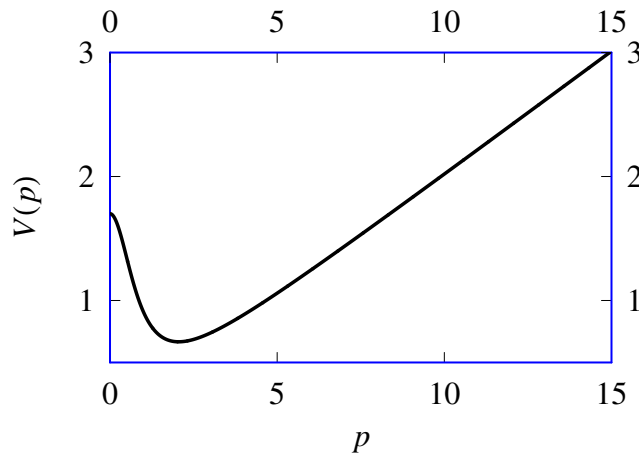
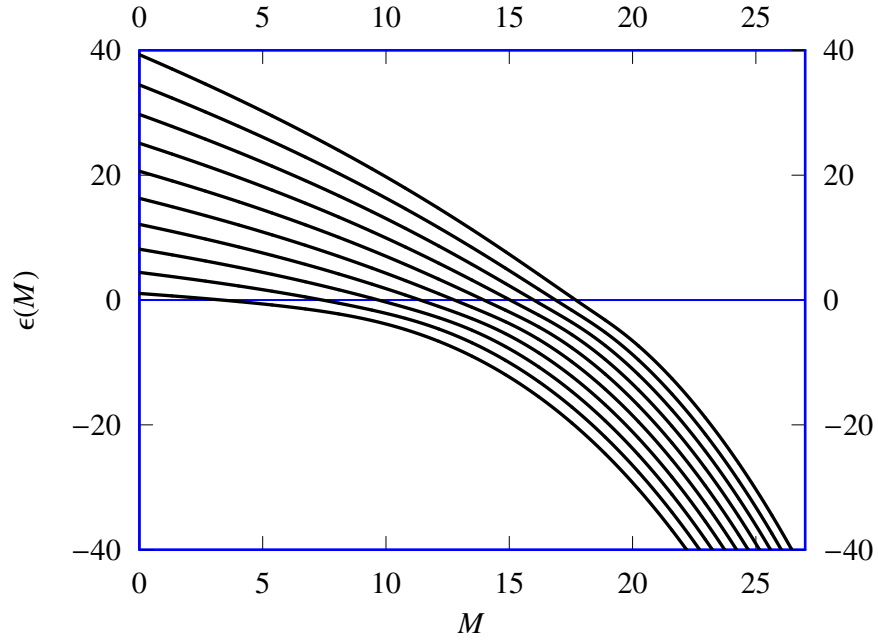
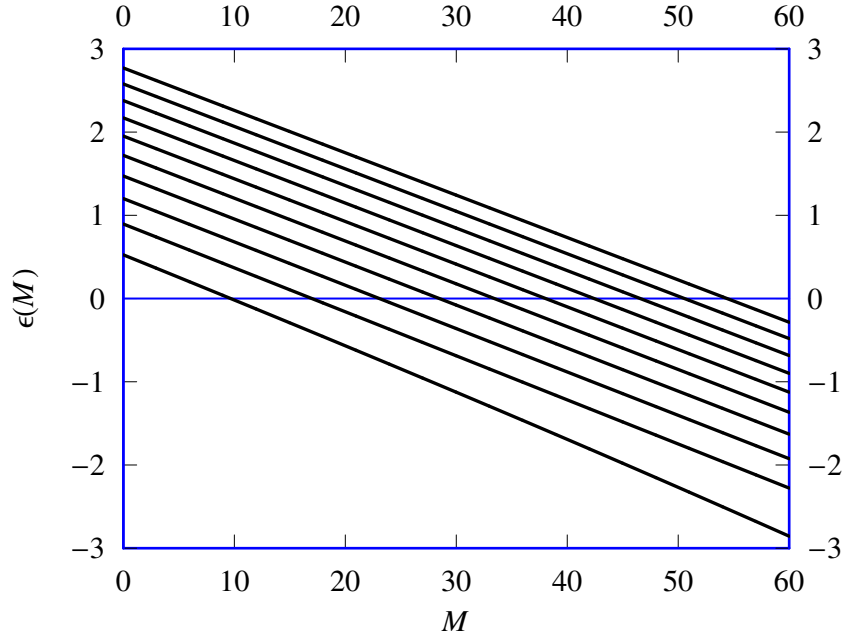


FIGURE 1. Effective potential $V(p)$ in the Hamiltonian \mathcal{H} for the harmonic-oscillator reduced Salpeter equation of time-component Lorentz-vector kernel $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, for $m = 1$ and $a = 10$ [arbitrary units].



(a)



(b)

FIGURE 2. Lowest-lying eigenvalues $\epsilon(M)$ of the auxiliary Hamiltonians $\mathcal{H} \equiv -D + U(p; M)$, derived from the reduced Salpeter equation (1) describing pseudoscalar (1S_0) fermion–antifermion bound states of constituents with mass $m = 1$, which experience a harmonic-oscillator interaction [$V(r) = ar^2$, $a = a^* \neq 0$] (a) of the Lorentz-scalar structure $\Gamma \otimes \Gamma = 1 \otimes 1$ (with the “binding” coupling constant $a = -10$) and (b) of the Lorentz-vector structure $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$ (with a “binding” coupling constant $a = 10$) [arbitrary units].

- For $m \neq 0$, because of the presence of the nasty differential operators $m^2 E^{-1} D E^{-1}$, the Hamiltonians \mathcal{H} in (2) corresponding to both Lorentz *scalar*, $\Gamma \otimes \Gamma = 1 \otimes 1$, and Lorentz *vector*, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, do *not* constitute (standard) Schrödinger operators. In these cases, however, by a suitable redefinition of the radial amplitudes $\phi(p)$, the (radial) differential equations may be transformed to eigenvalue equations of ($\ell = 0$) Schrödinger operators \mathcal{H} of the form $\mathcal{H} \equiv -D + U(p; M)$, which involve effective potentials $U(p; M)$ depending on the bound-state mass M as parameter. As might be guessed from the form of the corresponding Hamiltonian \mathcal{H} , for the Lorentz *scalar* the transformation required here simply reads $\phi(p) \rightarrow E \phi(p)$. For given M , and the appropriate sign of a , our effective potentials $U(p; M)$ are bounded from below and behave like $U(p; M) \rightarrow \infty$ for $p \rightarrow \infty$. Thus, the spectra of both “auxiliary” operators \mathcal{H} must consist entirely of *discrete* M -dependent eigenvalues $\varepsilon(M)$ (cf. Fig. 2). The derivatives of the latter eigenvalues with respect to M are, for all M , strictly definite. The bound-state masses M , defined by the zeros of the eigenvalues $\varepsilon(M)$, must then be discrete too. Since these eigenvalues $\varepsilon(M)$ are strictly *decreasing* functions of M a closer inspection establishes the bound-state masses M to be bounded from *below*.

In summary, given the semiboundedness of all the Hamiltonians \mathcal{H} entering in the radial equations and having established the discreteness of their spectra for harmonic-oscillator couplings of appropriate sign, our *harmonic-oscillator reduced Salpeter equation* poses, at least for a very wide class of Lorentz structures, a well-defined problem, with solutions describing *stable bound states* related to some *real discrete spectra bounded from below*.

GENERALIZATION TO THE (FULL) SALPETER EQUATION

It goes without saying that a similar study may be envisaged for the *full* Salpeter equation (as before, for bound-state constituents of equal mass in the rest frame of the bound state)

$$\Phi(\mathbf{p}) = \int \frac{d^3 q}{(2\pi)^3} \left(\frac{\Lambda^+(\mathbf{p}) \gamma_0 \sum_\Gamma V_\Gamma(\mathbf{p}, \mathbf{q}) \Gamma \Phi(\mathbf{q}) \Gamma \Lambda^-(\mathbf{p}) \gamma_0}{M - 2E} - \frac{\Lambda^-(\mathbf{p}) \gamma_0 \sum_\Gamma V_\Gamma(\mathbf{p}, \mathbf{q}) \Gamma \Phi(\mathbf{q}) \Gamma \Lambda^+(\mathbf{p}) \gamma_0}{M + 2E} \right). \quad (3)$$

There, however, the spectral analysis will be more complicated for the following reasons:

- Although the *squares* M^2 of all the mass eigenvalues M are guaranteed to be real [9] the spectrum is, in general, *not* necessarily real and, even in those cases where it can be shown to be real, it is certainly *not* bounded from below [9]. In particular, for the perhaps most important example of Bethe–Salpeter kernels involving only potential functions $V_\Gamma(\mathbf{p}, \mathbf{q})$ satisfying $V_\Gamma^*(\mathbf{p}, \mathbf{q}) = V_\Gamma(\mathbf{p}, \mathbf{q}) = V_\Gamma(\mathbf{q}, \mathbf{p})$ and coupling matrices Γ satisfying $\gamma_0 \Gamma^\dagger \gamma_0 = \pm \Gamma$ the spectrum of mass eigenvalues M in the complex- M plane consists of *real* opposite-sign pairs $(M, -M)$ and *imaginary* points $M = -M^*$.
- “Full-Salpeter amplitudes” $\Phi(\mathbf{p})$, that is, solutions of the full Salpeter equation (3), have more than one independent components. Thus, any *full* Salpeter equation with harmonic-oscillator interaction translates to a *system* of more than one second-order differential equations or, equivalently, a single differential equation of higher order.

The Salpeter amplitude for 1J_J states, e.g., involves two independent components, ϕ_1, ϕ_2 :

$$\Phi(\mathbf{p}) = \left[\phi_1(\mathbf{p}) \frac{\gamma_0(\boldsymbol{\gamma} \cdot \mathbf{p} + m)}{E} + \phi_2(\mathbf{p}) \right] \gamma_5 .$$

In particular, for interaction kernels of Lorentz-scalar ($\Gamma \otimes \Gamma = 1 \otimes 1$) or time-component Lorentz-vector ($\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$) structure, the *full* Salpeter equation (3), after getting rid of all angular variables, becomes equivalent to the fourth-order differential equation [10]

$$\left\{ 4E^2 - \frac{2a}{E} [\sigma(p^2 + 2m^2)D + pDp - 2] + \frac{a^2}{E} (m^2D + \sigma pDp - 2\sigma) \frac{1}{E}D \right\} \phi_2(p) = M^2 \phi_2(p) ,$$

with a sign factor σ (rendering possible a simultaneous study of both structures) given by

$$\sigma = \begin{cases} +1 & \text{for } \Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0 \quad (\text{time-component Lorentz-vector interactions}) , \\ -1 & \text{for } \Gamma \otimes \Gamma = 1 \otimes 1 \quad (\text{Lorentz-scalar interactions}) . \end{cases}$$

A notable exception is the Salpeter equation (3) with a harmonic-oscillator interaction of the BJK [8] Lorentz structure $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$: here one still arrives at a second-order differential equation that can be expressed in two equivalent ways [10],

$$4(E^2 - aDE) \phi_1(p) = M^2 \phi_1(p) \quad \text{or} \quad 4(E^2 - aED) \phi_2(p) = M^2 \phi_2(p) ,$$

since $2E \phi_1(p) = M \phi_2(p)$. Again, the spectrum of bound-state masses M is discrete [10].

ACKNOWLEDGMENTS

We wish to express our deepest gratitude to Bernhard Baumgartner, Harald Grosse and Heide Narnhofer for many interesting, stimulating, encouraging and helpful discussions.

REFERENCES

1. J. Parramore and J. Piekarewicz, *Nucl. Phys. A* **585**, 705 (1995) [nucl-th/9402019]; J. Parramore, H.-C. Jean, and J. Piekarewicz, *Phys. Rev. C* **53**, 2449 (1996) [nucl-th/9510024]; M. G. Olsson, S. Veseli, and K. Williams, *Phys. Rev. D* **52**, 5141 (1995) [hep-ph/9503477]; M. Uzzo and F. Gross, *Phys. Rev. C* **59**, 1009 (1999) [nucl-th/9808041].
2. A. B. Henriques, B. H. Kellett, and R. G. Moorhouse, *Phys. Lett. B* **64**, 85 (1976).
3. W. Lucha, K. Maung Maung, and F. F. Schöberl, *Phys. Rev. D* **63**, 056002 (2001) [hep-ph/0009185].
4. W. Lucha, K. Maung Maung, and F. F. Schöberl, in *Proceedings of the International Conference on "Quark Confinement and the Hadron Spectrum IV"*, edited by W. Lucha and K. Maung Maung, World Scientific, Singapore, 2002, p. 340 [hep-ph/0010078].
5. W. Lucha, K. Maung Maung, and F. F. Schöberl, *Phys. Rev. D* **64**, 036007 (2001) [hep-ph/0011235].
6. W. Lucha and F. F. Schöberl, *Int. J. Mod. Phys. A* **17**, 2233 (2002) [hep-ph/0109165].
7. J.-F. Lagaë, *Phys. Rev. D* **45**, 305 (1992); M. G. Olsson, S. Veseli, and K. Williams, *Phys. Rev. D* **53**, 504 (1996) [hep-ph/9504221].
8. M. Böhm, H. Joos, and M. Krammer, *Nucl. Phys. B* **51**, 397 (1973).
9. J. Resag, C. R. Münz, B. C. Metsch, and H. R. Petry, *Nucl. Phys. A* **578**, 397 (1994) [nucl-th/9307026].
10. Z.-F. Li, W. Lucha, and F. F. Schöberl, Vienna preprint HEPHY-PUB 843/07 (2007).